Dynamics of a sliding ladder leaning against a wall

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Abstract

This study is about the dynamics of a sliding ladder leaning against a vertical wall. The results are understood by considering the motion divided in two parts: (i) for $0 \le t \le t_s$ with one degree of freedom, and (ii) for $t > t_s$ with two degrees of freedom, where the separation is determined by the instance t_s, when the ladder loses its contact with the wall. The observed experimental details are explained by appealing a simple model based on elementary notions of mechanics. We emphasize some features, such as a maximum of the x component of the velocity and of the acceleration of the centre of mass in the first part, and a minimum of the normal reaction force on the floor in the second.

1. Introduction

The problem of a ladder leaning on a wall has been discussed in numerous introductory physics textbooks and in journals [1–8]. These works haved focused on the static equilibrium conditions of the ladder, considering friction against the wall and/or the floor. It has been shown that the reaction forces can be determined in quite a complex way, with the elastic properties of the ladder playing an important role. However, the dynamics of a sliding ladder has been studied less, and usually with the assumption that it is a rigid body that is not under friction conditions for both wall and floor surfaces simultaneously [7, 8].

Here we present a study, both experimental and theoretical, of the dynamics of the sliding ladder with friction at the end contacts with the wall and the floor. Video analysis of the ladder's motion was performed with *Tracker* [9, 10], freeware software that includes the module *Data Tool* for video modelling. Experimental configuration of this motion indicated that the wall and ground have the same coefficient of friction and that the ladder is, initially, just on the verge of slipping downwards.

2. Theory

The model we use relies on the assumption that the ladder is a homogeneous beam-shaped rigid body of length L and mass m. The coefficient of friction with the wall and the floor is μ. The angle of the ladder with the floor (θ) was chosen as the variable to describe the movement while the ladder is in contact with the wall (figure 1). The motion of the ladder is governed by the equations of mechanics for rigid body motion, namely:

$$\vec{R} = m\,\ddot{\vec{r}} \qquad (1)$$

$$\vec{M}_{CM} = I_{CM}\,\ddot{\vec{\theta}}\,, \qquad (2)$$

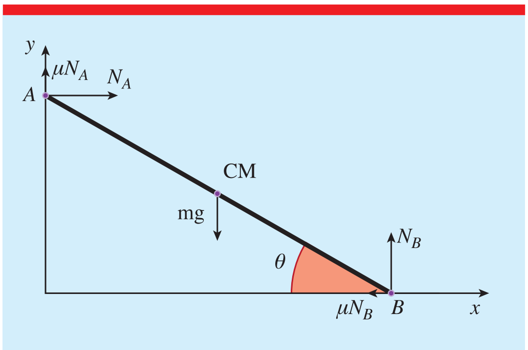

Figure 1. The scheme of the ladder as a homogeneous beam-shaped rigid body. N_A and N_B are the normal reactions in A and B, respectively.

where $\vec{\ddot{r}}$ is the acceleration of the centre of mass (CM), CM = (x,y), $\vec{M}_{CM}$ is the resultant torque of the external forces about CM, I_{CM} is the moment of inertia about CM assumed as $1/12\, mL^2$, and $\ddot{\theta}$ is the angular acceleration of the ladder during its intrinsic rotation. $\vec{R}$ is the vector sum of the external forces $\vec{F_A} = (N_A,\ \mu N_A)$, $\vec{F_B} = (-\mu N_B,\ N_B)$ and weight $m\vec{g}$, considering $\mu_{\text{wall}} = \mu_{\text{floor}} = \mu$, and therefore the normal reactions in A and B are respectively N_A and N_B. The ladder is in equilibrium for an angle larger than the critical angle $\theta_c = \tan^{-1}(1-\mu^2)/2\mu$.

While there is contact with the vertical wall ($\dot{y} = x\ \dot{\theta}$), equations (1) and (2) can be rewritten as:

$$\ddot{x} = \frac{1}{m}\,(-\mu N_B + N_A) \tag{3}$$

$$\ddot{y} = \frac{1}{m}\,(\mu N_A + N_B - mg) \tag{4}$$

$$\ddot{\theta} = \frac{6}{mL}\,[\sin\theta(\mu N_B + N_A) + \cos\theta(\mu N_A - N_B)] \tag{5}$$

Thus, the following expressions of dynamics are obtained:

$$\ddot{\theta} = -\frac{3}{L}\frac{2\mu g\sin\theta - g\cos\theta + \mu^2 g\cos\theta - \mu L(\dot{\theta})^2}{\mu^2-2} \tag{6}$$

$$N_A = -\frac{m}{2(\mu^2-2)}[3g\sin\theta\cos\theta + L\mu\sin\theta(\dot{\theta})^2 - 2L\cos\theta(\dot{\theta})^2 - 2\mu g + 3\mu g(\cos\theta)^2] \tag{7}$$

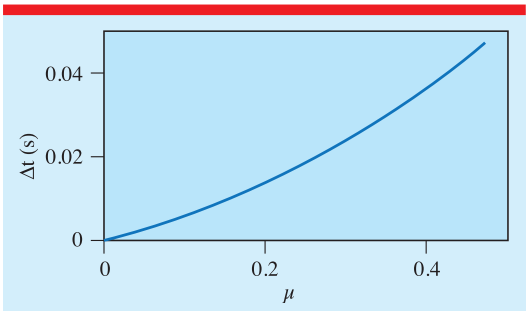

Figure 2. The difference $\Delta t = t_s - t_m$ as a function of the coefficient of friction (computational results).

$$N_B = \frac{m}{2(\mu^2-2)}[-3\mu g\sin\theta\cos\theta + 2L\sin\theta(\dot{\theta})^2 + L\mu\cos\theta(\dot{\theta})^2 - 4g + 3g(\cos\theta)^2]. \tag{8}$$

The ladder loses contact with the vertical wall at time t_s, corresponding to $N_A = 0$ at an angle θ_s given by

$$\frac{L}{g}[\mu\sin\theta_s - 2\cos\theta_s](\dot{\theta}(t_s))^2 + 3\mu(\cos\theta_s)^2 + 3\sin\theta_s\cos\theta_s = 2\mu. \tag{9}$$

It is easy to prove that if $\mu > 0$ the ladder achieves the maximal speed of x component at the time t_m, before losing contact with the wall ($t_m < t_s$). In the case of no friction ($\mu = 0$), that maximum value occurs exactly for $t_m = t_s$ [3] (figure 2).

After losing contact with the vertical wall ($t > t_s$), the beam's motion has two degrees of freedom (θ and x). As a consequence, equations (1) and (2) result in the following:

$$\ddot{\theta} = \frac{3[2\mu g\sin\theta + L\sin\theta\cos\theta(\dot{\theta})^2 - \mu L(\sin\theta)^2(\dot{\theta})^2 - 2g\cos\theta]}{L[1+3(\cos\theta)^2 - 3\mu\cos\theta\sin\theta]} \tag{10}$$

$$\ddot{x} = \frac{\mu(2g - L\sin\theta\ \dot{\theta})}{2[3\mu\cos\theta\sin\theta - 3(\cos\theta)^2 - 1]} \tag{11}$$

$$N_A = 0 \tag{12}$$

$$N_B = \frac{1}{2}\frac{m(2g - L\sin\theta(\dot{\theta})^2)}{1+3(\cos\theta)^2 - 3\mu\cos\theta\sin\theta} \tag{13}$$

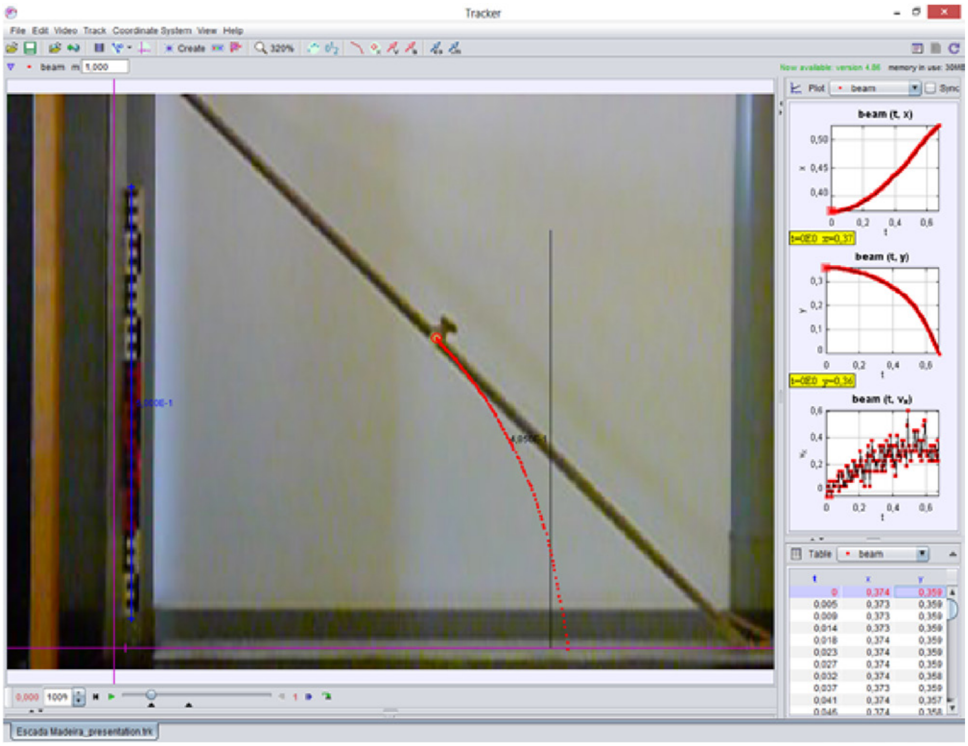

Figure 3. Experimental setup of a sliding ladder leaning against a wall (screenshot of Tracker's main window). The coordinates of the CM during the motion are represented by red points. $x(t)$, $y(t)$ and $\dot{x}(t)$ are also represented.

These equations allow us to recognize a discontinuity in $\dot{N}_A$ and $\dot{N}_B$ at $t = t_s$. Therefore, there is also a discontinuity in the third derivatives of the variables that describe the ladder's motion, namely x, y and θ (i.e. in the derivatives of linear and angular acceleration). This result is represented in the computational curves of figures 6 and 7 and will be detailed in the following sections.

3. Experimental and computational results

An experiment was performed using a homogeneous rigid beam representation of a ladder, with a mass of 0.23572 kg and a length of 1.005 m. We used a high-speed digital photography acquisition camera (Panasonic FZ 100 LUMIX, 220 frames s^{-1}) to record the motion of the beam's centre of gravity. The experimental setup is embedded in figure 3, which is a screenshot of Tracker's main window. The coordinates of the CM during the beam's motion (represented by red points) and $x(t)$, $y(t)$ and $\dot{x}(t)$ are depicted.

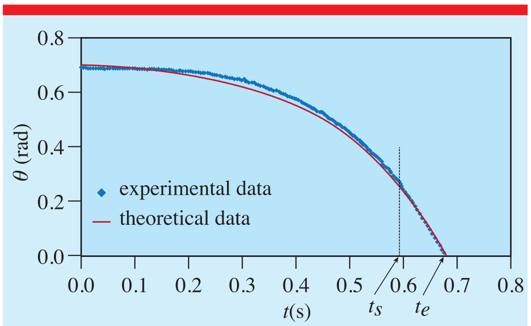

Figure 4. The angle θ of inclination of the ladder as a function of time. Representation of theoretical data (full red line) and experimental data (rhombus blue points). The vertical dot line marks the instant t_s when the beam loses contact with the wall, while t_e is the instant of full stop of the ladder's motion.

Initially, the beam was placed against the wall at an angle $\theta_o = 0.69$ rad, slightly lower than the experimental critical angle (the angle at which the beam starts to fall), and then left free to move.

Figure 4 shows the time variation of θ for both experimental and computational studies.

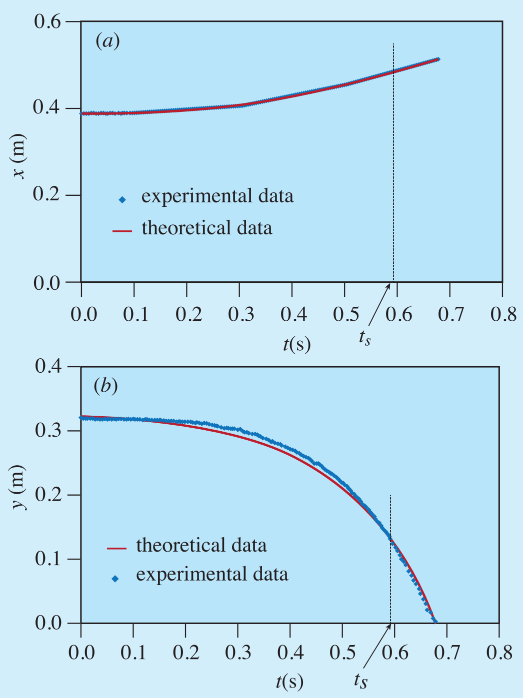

Figure 5. Theoretical and experimental results for the coordinates of CM as a function of time. The vertical dot line marks the instant t_s when the beam loses contact with the wall, while t_e is the instant of full stop of the ladder's motion.

The least-squares criterion was applied to adjust computational data to experimental and estimate the coefficient of friction. It is evident from the plot that the agreement is fairly good, not only during the motion of the beam against the wall ($t < t_s$, where t_s = 0.592574 s is a numerical result and is identified in figure 4 by a vertical dot line), but also during the interval between losing contact and the full stop (at t_e = 0.6791960 s, also obtained with computational analysis). The value obtained for μ from the curve fit was 0.43.

The evolution of coordinates x and y of the CM during the motion (figures 5(*a*) and (*b*)) also confirms the reasonable good performance of the model assumed.

The slight deviations observed from comparison of the experimental data and the computational data are possibly due to the assumption of an ideal beam for the ladder, a constant value for the coefficient of friction, and experimental errors on measuring the coordinates of CM.

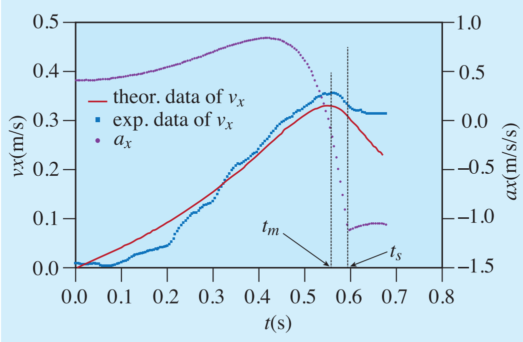

Figure 6. Representation of the evolution with time of the components of the velocity (experimental and theoretical) and acceleration of the CM along the x direction. Note that the left axis corresponds to speed, while the right axis corresponds to acceleration. The vertical dot line marks the instant t_m where v_x is maximal, while t_s is the instant when the beam loses contact with the wall.

As we suggested earlier, according to the model the x component of the speed should attain a maximum at t_m, before the beam loses contact with the wall. In this case, we have obtained t_m = 0.551314 s from computational analysis. Experimental and computational data match within experimental uncertainty, as shown in figure 6.

We can also disclose a discontinuity in the derivative of the horizontal component of the acceleration at t_s, the instant when the ladder loses contact with the wall. This is related to the discontinuity in the derivative of the net force along the x direction, as supported by equations (12) and (13).

The existence of the maximum of $\dot{x}$ before t_s can be easily understood, since the normal forces N_A and N_B are not constant during the motion (figure 7), which is not always an obvious observation. From equation (3), we conclude that the maximum occurs when $\mu N_B = N_A$.

The plot of the acceleration along the x direction (figure 6) reveals additional features of the beam's motion. Again, the existence of the maximum of $\ddot{x}$ before t_s can be supported using mathematical analysis, looking for the solution of the equation $\mu\, \dot{N}_B = \dot{N}_A$.

We also note that $\dot{F}_B$ is discontinuous at t_s and exhibits a minimal value within the interval $[t_s,\ t_e]$. This outcome anticipates a careful and more involved analysis of the dynamics of a ladder after losing contact with the wall, to be published later.

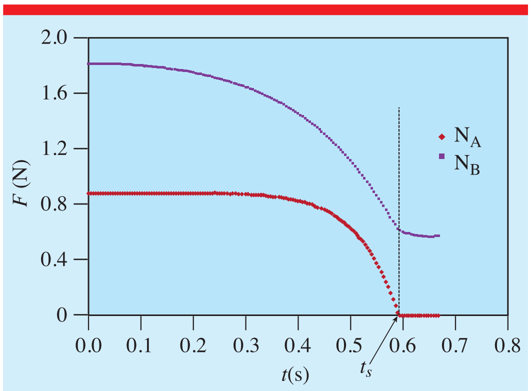

Figure 7. Normal components of the reaction forces at the ends of the ladder (points A and B) as a function of time. These forces point in perpendicular directions, as specified in figure 1. The vertical dot line marks the instant t_s when the beam loses contact with the wall. Note that both $\dot{N}_A$ and $\dot{N}_B$ are discontinuous at t_s.

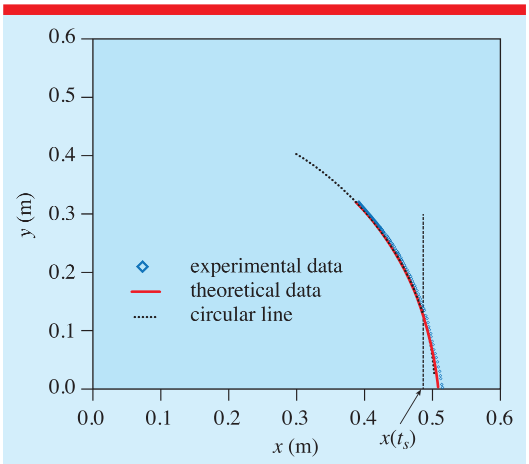

Figure 8. The trajectory of the CM. The vertical dot line reveals the position of the CM at the instant t_s when the ladder loses contact with the wall. After t_s the trajectory of the CM is no longer a circular line.

The trajectory of the CM is represented in figure 8 both experimentally and theoretically. While there is no doubt about the geometric nature of this Cartesian line for $t < t_s$ (circle line), no easy conclusion can be reached for $t > t_s$, which reinforces the need for further study of the dynamics of the last part of the movement.

4. Conclusion

This work addresses the dynamics of a sliding ladder in a context that, as far as we know, has never been studied previously. We restrict our study to the case where coefficients of friction along the two surfaces are equal, although no significant modifications should result if the coefficients are different or tend to zero.

The fact that the ladder leaves the wall at a specific instant t_s when the normal reaction force on the wall N_A vanishes, before reaching the end of the motion, indicates the importance of considering the falling motion in two parts: for $0 \leq t \leq t_s$ with one degree of freedom and for $t > t_s$ with two degrees of freedom. This second part of the motion has been ignored so far in the literature [2, 7, 11].

The physical interpretation of the dynamics of the ladder is more complex than is usually admitted in the literature [12]. In particular, the normal reaction forces change along the motion, resulting in surprising observations such as, for example, the maximum for the speed (supported by experimental data) along the x direction, and the discontinuity of the time derivatives $\dot{N}_A$ and $\dot{N}_B$ at the instant when the ladder loses contact with the wall. A more general discussion of the problem is in progress and will be published later.

Finally, we stress that this problem provides an excellent introduction to the modelling process for college students and physics and mathematics teachers. Students have the opportunity to practise experimental techniques, theoretical modelling and video data acquisition and analysis. Additionally, the problem is a challenge where students can discuss their ideas with their fellow students and their teachers, creating a rich environment for a better understanding of physical science.

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