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An Inexpensive Technique to Measure Coefficients of Friction with Rolling Solids

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Frictional forces are everywhere. They are probably among the most important macroscopic forces in our daily routine because we depend on them to walk, to pick up objects, and even to eat! They affect our lives in different ways, and thus it is very important to understand them better; this is what we say to our students. There are many experimental methods and techniques to measure static and kinetic frictional forces (and to determine their corresponding coefficients of friction), which we can find in several textbooks.[1-5] Most of them involve pulling/pushing blocks along a flat surface and expensive equipment for accurate measuring (sensors, computers, etc.). In this paper we show how to find both coefficients of friction, static and kinetic, with rolling objects instead of blocks in a very simple way and using nonexpensive laboratory equipment.

Theory

Suppose we put a solid cylinder of radius R and mass M on an incline and let it freely rotate along it, as shown in Fig. 1, where we have also represented the forces acting on the cylinder (this is very useful in the classroom and students should always be encouraged to do it).

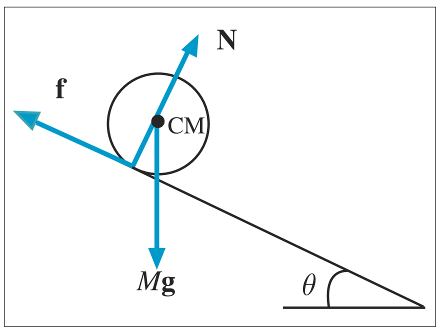

Fig. 1. Schematic representation of the forces acting on a cylinder rolling freely along an incline.

The description of the cylinder's motion can be done mathematically by using both fundamental equations, for translation

$$\sum \mathbf{F}_{\text{ext}} = M\,\mathbf{a}_{\text{CM}}\,, \tag{1}$$

and for rotation

$$\sum \boldsymbol{\tau}_{\text{CM}} = I_{\text{CM}}\,\boldsymbol{\alpha}\,, \tag{2}$$

where $I_{\text{CM}} = \frac{1}{2}MR^2$ is the moment of inertia about the axis of the cylinder, and $\Sigma\boldsymbol{\tau}_{\text{CM}}$ is the sum of the torques about the center of mass.

If the cylinder rolls without slipping, then $a_{\text{CM}} = R\alpha$. Therefore, Eqs. (1) and (2) become, respectively, $Mg\sin\theta - f = Ma_{\text{CM}}$ and $fR = I_{\text{CM}}(a_{\text{CM}}/R)$. Combining these equations, we find expressions for the static frictional force and acceleration of the CM:

$$f = \frac{1}{3}Mg\sin\theta; \quad a_{\text{CM}} = \frac{2}{3}g\sin\theta. \tag{3}$$

Here it must be stressed that f is always less (or equal) than its maximum value, $f_{\max} = \mu_s Mg\cos\theta$ (μ_s is the coefficient of static friction), i.e., for every angle θ satisfying the condition $\tan\theta < 3\mu_s$, the cylinder rolls without slipping; when $\tan\theta > 3\mu_s$, it rolls and slips. Therefore, there is a critical angle θ_c for which the coefficient μ_s can be determined:

$$\mu_s = \frac{1}{3}\tan\theta_c. \tag{4}$$

Now let us suppose the angle θ in Fig. 1 is higher than θ_c. The cylinder rolls and slips along the incline, and the forces acting on the cylinder are the same as those represented in the figure, but f is now given by $f_k = \mu_k Mg\cos\theta$ (μ_k is the coefficient of kinetic friction). Making use of Eqs. (1) and (2), we get the analytical expressions for the angular and linear accelerations:

$$\alpha = \frac{2\mu_k g\cos\theta}{R}; \tag{5}$$
$$a_{\text{CM}} = g(\sin\theta - \mu_k\cos\theta).$$

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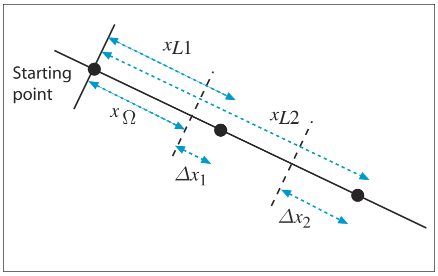

Fig. 2. Representation of the positions marked by the cylinder when it completes each rotation.

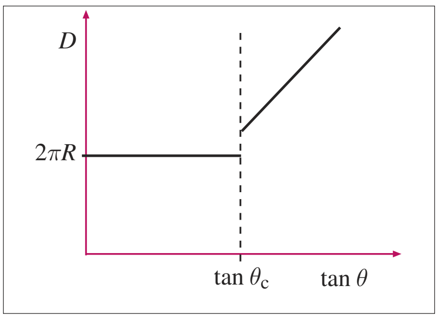

Fig. 3. Graph of the distance between two successive marks on the incline vs tan θ, according to Eq. (11). The jump at θ_c is proportional to $\left(\frac{1}{\mu_k}-\frac{1}{\mu_c}\right)$; therefore it vanishes when μ_k is close to μ_c.

Since the cylinder starts from rest at position $x = 0$, the position of the CM at instant t is given by:

$$x_L = \frac{1}{2}a_{CM}t^2 = \frac{1}{2}g(\sin\theta - \mu_k\cos\theta)t^2. \quad (6)$$

Meanwhile, the cylinder rotates by an angle Ω,

$$\Omega = \frac{1}{2}\alpha t^2 = \frac{\mu_k g\cos\theta}{R}t^2. \quad (7)$$

As it rolls and slips, x_L does not match the distance $x_\Omega = \Omega R$ along the circular arc described by Ω, i.e., there is a difference Δx between the displacement of the CM and the length along the arc described by any point at the external surface of the cylinder,

$$\Delta x = x_L - x_\Omega = \frac{1}{2}g(\sin\theta - 3\mu_k\cos\theta)t^2. \quad (8)$$

The time elapsed for N complete rotations ($\Omega = 2N\pi R$) can be determined from Eq. (7),

$$t_N = \left[\frac{2N\pi R}{\mu_k g\cos\theta}\right]^{\frac{1}{2}}. \quad (9)$$

Therefore, the difference Δx_N for N complete rotations is given by

$$\Delta x_N = N\pi R\left(\frac{\tan\theta}{\mu_k} - 3\right). \quad (10)$$

If we could mark the positions where the cylinder completes each rotation on the surface of the incline, we would get a diagram like the one in Fig. 2. The distance between two successive marks,

$$D = x_{L(N+1)} - x_{LN} = 2\pi R + \pi R\left(\frac{\tan\theta}{\mu_k} - 3\right), \quad (11)$$

could therefore be measured.

By plotting D as a function of tan θ, we would expect to obtain an experimental graph similar to that represented in Fig. 3, which was determined by simulation. Above the critical angle θ_c the graph has a constant slope m, which allows us to determine the coefficient of kinetic friction by using Eq. (11):

$$\mu_k = \frac{\pi R}{m}. \quad (12)$$

The coefficient of static friction can be easily calculated from the critical angle,

$$\mu_s = \frac{\tan\theta_c}{3}. \quad (13)$$

Experimental procedure

All you need for this experiment is a flat inclined surface, a protractor, a solid cylinder (we have used one with 44.5-mm diameter and 47.0-mm length), a small brush, a measuring tape and some ink (permanent ink will be just fine). Glue the brush handle to one of the cylinder's bases so that the hair stands just a bit outside the border (see Fig. 4). Choose a value θ for the slope of the incline and measure it with the protractor. Drop some ink into the brush and put the cylinder at the top of the incline. Carefully rotate the cylinder until the hair almost touches the surface. By doing this, you will get the first mark on the surface when the cylinder starts to move. You are now ready to start your experiment (see Fig. 5).

Let the cylinder roll down freely. Measure the distance between the marks on the surface and repeat this procedure for different angles. As you will see for yourself, if the bases of the cylinder do not have exactly the same radius it will describe a curved line instead of a straight line, so the distance between marks will not be constant—this is much more important when the cylinder slips; if this is the case, we suggest you simply measure the distance between the two first marks. Do not forget to clean the surface before each sampling!

In our own experiment, we used a common glass surface and a polished cylinder made of aluminium. The results are plotted in Fig. 6.

Above 22° the slope in the experimental graph witnesses the slipping of the cylinder while rotating down the incline. Computing the slope of

Fig. 4. One of the bases of the aluminium cylinder. The brush handle is glued in such a way that the hair stands just a bit outside the border.

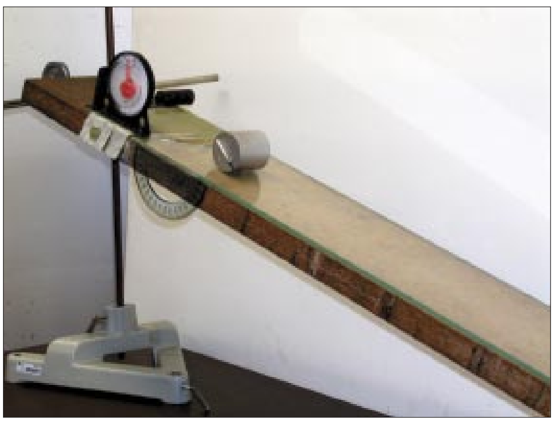

Fig. 5. Experimental setup used. The cylinder is ready to go along the incline.

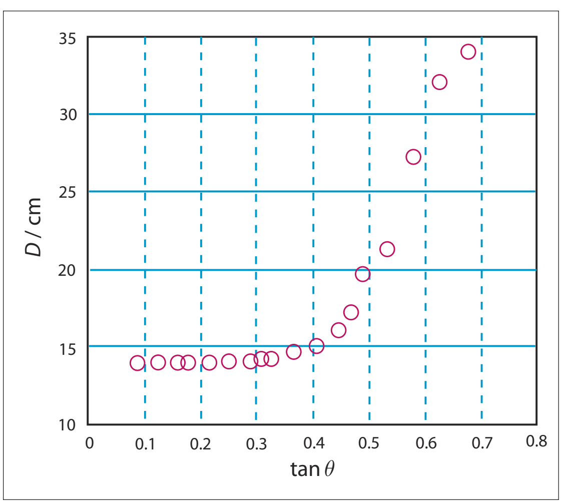

Fig. 6. Distance between the two first successive marks made by a polished aluminium cylinder on a glass incline vs tan θ, obtained from experimental data.

this graph and using Eqs. (11) and (12), we get as a result for the coefficient of kinetic friction (aluminum on glass) $\mu_k = 0.091 \pm 0.007$, the accuracy being affected essentially by the uncertainty of the slope.

As the cylinder effectively starts to slip above the critical angle of 20°, the coefficient of static friction (aluminum on glass) can be determined from Eq. (13), and so we have come to $\mu_s = 0.121 \pm 0.007$, where the accuracy is affected only by the uncertainty of the critical angle.

Between 20° and 22° we observed a nonpredicted behavior that we assign to a transient regime where both static and kinetic friction are present, i.e., at some points the cylinder rolls and slips, and at others it only rolls. Such a situation could be found in other experimental measurements, so teachers and students should be aware of it.

Conclusion

We have shown in a very easy and inexpensive way how to measure the coefficients of static and kinetic friction with rolling objects. This method can be used in any middle school laboratory lacking equipment such as sensors and computers; you just need a small brush, a measuring tape, and some ink. Results are easily compared with the theory, but we recommend that Eqs. (10) and (11) be conceptually discussed in the classroom and interpreted in terms of the cylinder's motion (simultaneous rolling and slipping, angular and linear speed in each instant t_N, and so on).

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