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# The Centre of Mass of a "flying" Body Revealed by a Computational Model

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## Abstract

The interpretation of complex trajectories of rigid bodies by the identification of their centre of mass (CM), has a large potential for improving the understanding of the CM's concept at college and university level. Therefore, it is not surprising that there are several techniques described in the literature concerning how to identify the centre of mass (CM) of rigid bodies. However, these techniques fail when the CM's position in the body's frame of reference changes when the body is at motion.

Is this work we present a computational model that allows the identification of the CM with a very good accuracy, either when the CM's position changes or is fixed in the body's frame of reference. This model can be used for a system of bodies moving in a plane, for which the CM of each body coincides with its geometric centre.

The effectiveness of this model is tested with experiments using video acquisition and numerical analysis, and can be done in experimental classes under controlled conditions. Students are then able to compare the computed CM with the experimental CM, and investigate why the bodies sometimes present weird trajectories. This property applies especially in sports, so the model can be also very useful as an educational resource for the explanation of the motion of athletes, namely as a tool for optimizing their performances.

**Key Words:** Centre of mass; athletes' motion; video analysis; Science and Art**.**

## Introduction

The interpretation of the trajectories of rigid bodies by the identification of their centre of mass (CM), allows students to understand that the CM's concept is more than a formal definition. It represents how the total mass of the body reacts to external forces and also allows students to understand that the trajectory of the CM is independent of how the body rotates due to external or internal forces.

The centre of mass (CM) of a body is known to be coincident, in most cases, with the centre of gravity [1]. This particularity enabled the development of different techniques for identifying the position of the CM, either by static setups [1-4], numerically [3] and by kinematics experiments [5-6].

In all those cases the CM's position in the body's frame of reference does not change with time. However, if the position of the CM is forced to move by internal forces during the body's motion, the traditional methods fail to identify the CM. Several attempts to overcome this issue can be found in the literature, many of them related to multibody robotic systems where computational methods were applied to determine the displacement of the CM of an internal moving platform, as can be found in reference [7] and references within.

In this work we present a computational model that is based on the kinetic analysis of the body's motion and on video recording, which provides a solution to the problem. Some experimental examples, which can be explored by college or university students, are given to test the efficacy of the model.

## The model

Let's suppose we have a rigid body that is tossed as a projectile in a vertical plane, with translational and rotational speeds. For simplicity, we will admit that friction between the body and the air can be neglected, as well as buoyancy forces.
As we know, in the laboratory (inertial) frame of reference, the $x$ component of the body's CM trajectory can be described as a function of time by a linear equation, while the $y$ component of the body's CM trajectory can be described as a function of time by a parabolic equation.
Supposing we have $N$ experimental data points of this trajectory, they can be fitted by the following mathematical functions

$$f_x(t) = At + B \tag{1}$$

$$f_y(t) = at^2 + bt + c \tag{2}$$

where $A$, $B$, $a$, $b$ and $c$ are constants. The *least square minimum* method stands that all these parameters are computed by minimizing the sums $G_x$ and $G_y$ of the square differences between the experimental data and the fitted data, defined as

$$G_x = \sum_{i=1}^{N} [f_x(t_i) - x(t_i)]^2 \tag{3}$$

$$G_y = \sum_{i=1}^{N} [f_y(t_i) - y(t_i)]^2 \tag{4}$$

where $x(t_i)$ and $y(t_i)$ are the experimental data at instant $t_i$ and consequently, the total sum

$$G = G_x + G_y \tag{5}$$

is also minimized.

For an extensive body in motion it is difficult to track directly its CM, although we can still track distinct parts of the body.

When the position of the CM is fixed in the body´s frame of reference, the distance between each body´s part and the CM is constant. However, when a portion of mass of the body is dislocated due to internal forces, the position of the centre of mass changes in the body's frame of reference and therefore, the relative positions between the CM and each body's parts are no longer constant. In this work we introduce some changes in the algorithms of the least square minimum method to estimates the CM trajectory by following the trajectory of different body's parts.

So, instead of one set of experimental data (or trajectory) from the body, we shall consider $W$ sets of trajectories each one with the same $N$ data points ($N$ instants $t_i$). We also admit that each set corresponds to a $j$ position on the body, which remain unchanged relatively to one each other in the body's frame of reference. These $j$ positions are within a region where we estimate the CM is located. All experimental data will be fitted by the mathematical functions (1) and (2).

At each instant $t_i$, we can admit that pairs $(x_j, y_j)(t_i)$ from each $j$ set of data contributes to the square differences $[f_x(t_i) - x_j(t_i)]^2$ and $[f_y(t_i) - y_j(t_i)]^2$ differently, because some data pairs may be located closer to the CM's position than others. Thus we must include in the model a weight parameter $p_{ij}$, to each $j$ pair data and for every instant $t_i$. There are two ways to do this: either $p_{ij}$ multiplies the square difference $[f_x(t_i) - x_j(t_i)]^2$, or $p_{ij}$ multiplies the difference $[f_x(t_i) - x_j(t_i)]$. Preliminary studies obtained by us suggest numerically better results when the square differences include the parameter $p_{ij}$, as follows

$$g_x(t_i) = \sum_{j=1}^{W} \{p_{ij}[f_x(t_i) - x_j(t_i)]\}^2 \tag{6}$$

$$g_y(t_i) = \sum_{j=1}^{W} \{p_{ij}[f_y(t_i) - y_j(t_i)]\}^2 \tag{7}$$

with the condition that for each instant $t_i$, $\sum_{j=1}^{W} p_{ij} = 1$ (for example, if one $j$ pair $(x_j, y_j)(t_i)$ matches exactly the fit equations, then the corresponding $p_{ij} = 1$ and for all $k \neq j$, $p_{ik} = 0$).

The sum of the weighted square differences to be minimized in $x$ and $y$ component are, therefore,

$$g_x = \sum_{i=1}^{N} g_x(t_i) = \sum_{i=1}^{N} \sum_{j=1}^{W} \{p_{ij}[f_x(t_i) - x_j(t_i)]\}^2 \tag{8}$$

$$g_y = \sum_{i=1}^{N} g_y(t_i) = \sum_{i=1}^{N} \sum_{j=1}^{W} \{p_{ij}[f_y(t_i) - y_j(t_i)]\}^2 \tag{9}$$

and consequently, the total sum of weighted square differences is

$$g = g_x + g_y \tag{10}$$

The calculus of this model implies $W$x$N$ weight parameters $p$ to be fit in addition to $a$, $b$, $c$, $A$ and $B$ constants, in order to compute the parabolic functions $f_x(t)$ and $f_y(t)$ that best adjust to the whole experimental data.

Figure 1 presents a simplified diagram of the whole algorithm of the model.

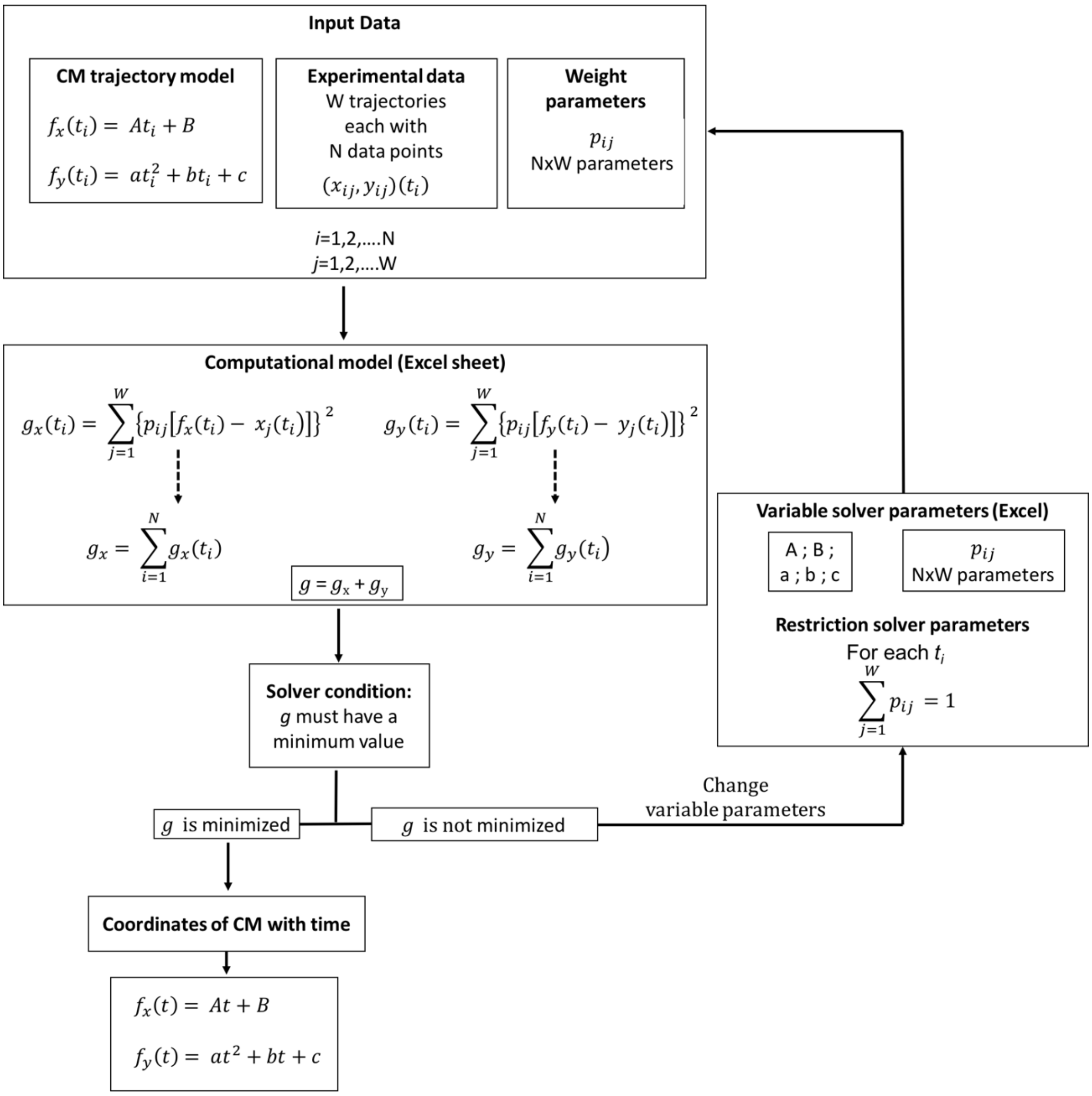


Figure 1: simplified diagram of the whole algorithm of the computational model, as described in the text.

## Experimental

To test our model, we used data from an experiment, where several colour marks on a tubular rigid body with 36 cm length (figure 2) were followed by video recording [8-9]. The mass of the tube was 106,0 g.

In the first experiment, the position of the CM does not change with time in the body's frame of reference. In the second experiment, a spring (of insignificant mass) with a small metallic ball (of mass 55,5 g) are put inside the tube, making the CM of the body change with time in the body's frame of reference. In both experiments, the position of the CM is fairly well calculated as the distribution of masses within the body is well identified.

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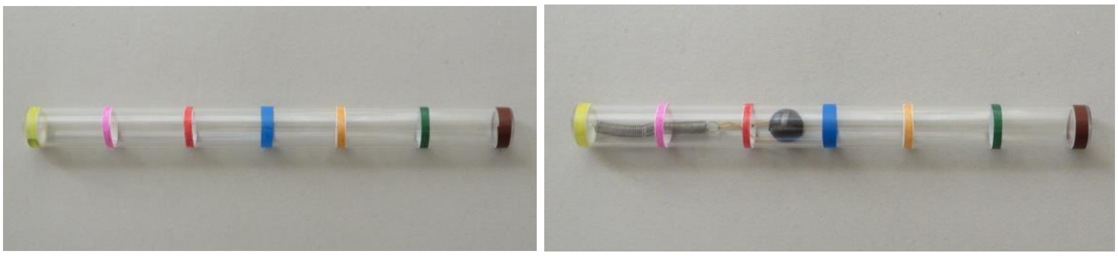

Figure 2: colour markers distributed on a transparent tubular rigid body. In the right image, a spring with a small metallic ball is introduced inside the tube, to change the position of the CM with time in the body's frame of reference during the motion.

The videos were captured by a digital photo camera Canon SX270 HS with 640 x 480 pixel resolution at 120 frames per second. In order to quantitatively characterize the motion, a calibrated reference bar was included in the video recording. The accuracy in the video image calibration is typically 2 pixels, corresponding to an estimated error of about 0.8 mm. Data acquisition for the position of each marker has a maximal estimated accuracy of about 2 mm due to some blurred images during the motion.

Experiment 1:

In this experiment, we made an oblique throw of the tube with rotation [10]. Figure 3 is a snapshot of Tracker software main window and shows the trajectories of the colour markers separated by 6 cm from each others, as well as the trajectory of the body's CM computed by the software.

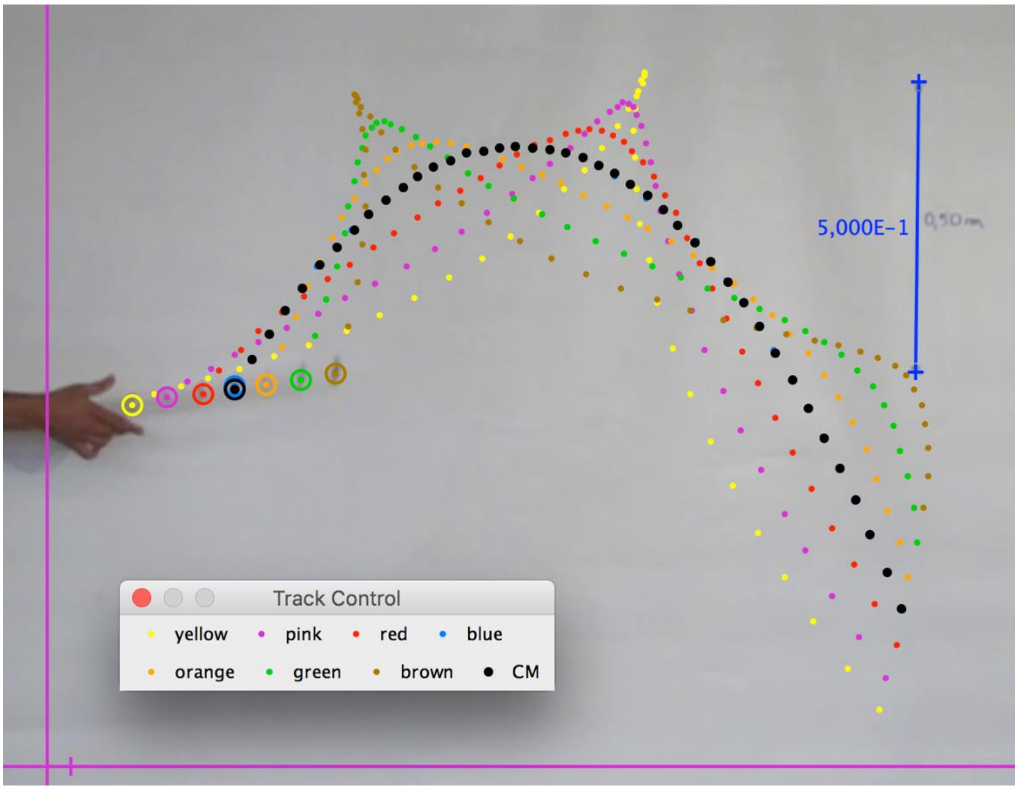


Figure 3: Trajectories of the colour markers of the body. The black dots correspond to the experimental CM as computed by Tracker software using the know mass of the tube.

The computational model was then used to fit the experimental data and find the position of the CM. Figure 4 shows the experimental *x* and *y* positions of the CM (from Tracker software) and the corresponding computed CM positions (curve fit from the model) with time. We can see that the agreement between the experimental and the computed data is excellent, and the average difference between them in $x$ and $y$ coordinates is 2.3 mm and 1.6 mm, respectively.

The results show that the model is adequate for identifying the CM in the described experimental conditions.

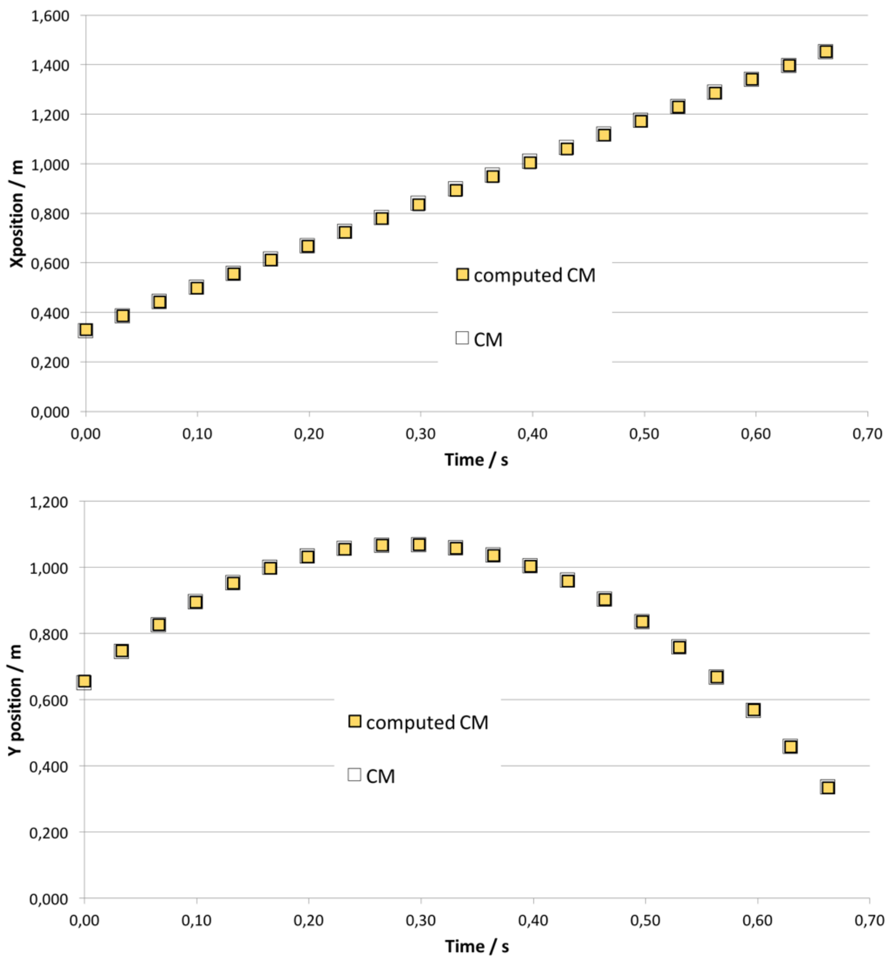


Figure 4: $x$ and $y$ positions with time, of the computed CM (curve fit) from the model and the experimental CM (from Tracker). The match is excellent.

Face to the good results in testing the model, we turn now out attention to the second experiment.

<u>Experiment 2</u>:

Introducing the spring and the metallic ball, we know that the CM of the body is going to change in the body's frame of reference when thrown obliquely with rotation [11]. Therefore, we will have to apply our fitting model to compute the CM in the laboratory frame of reference.

Figure 5 shows a snapshot of Tracker software main window with the trajectories of the colour markers and the metallic ball, as well as the trajectory of the body's CM computed by the software. We can now see the trajectories of all markers are quite different from those presented in figure 3, except the CM that presents the characteristic parabolic shape.

For the computation of the CM from our model we had to fit 161 $p$ parameters in order to obtain constants $A$, $B$, $a$, $b$ and $c$ of the fitting equations (1) and (2), by minimizing the resulting value of equation (10). The model was implemented using a MSExcel worksheet.

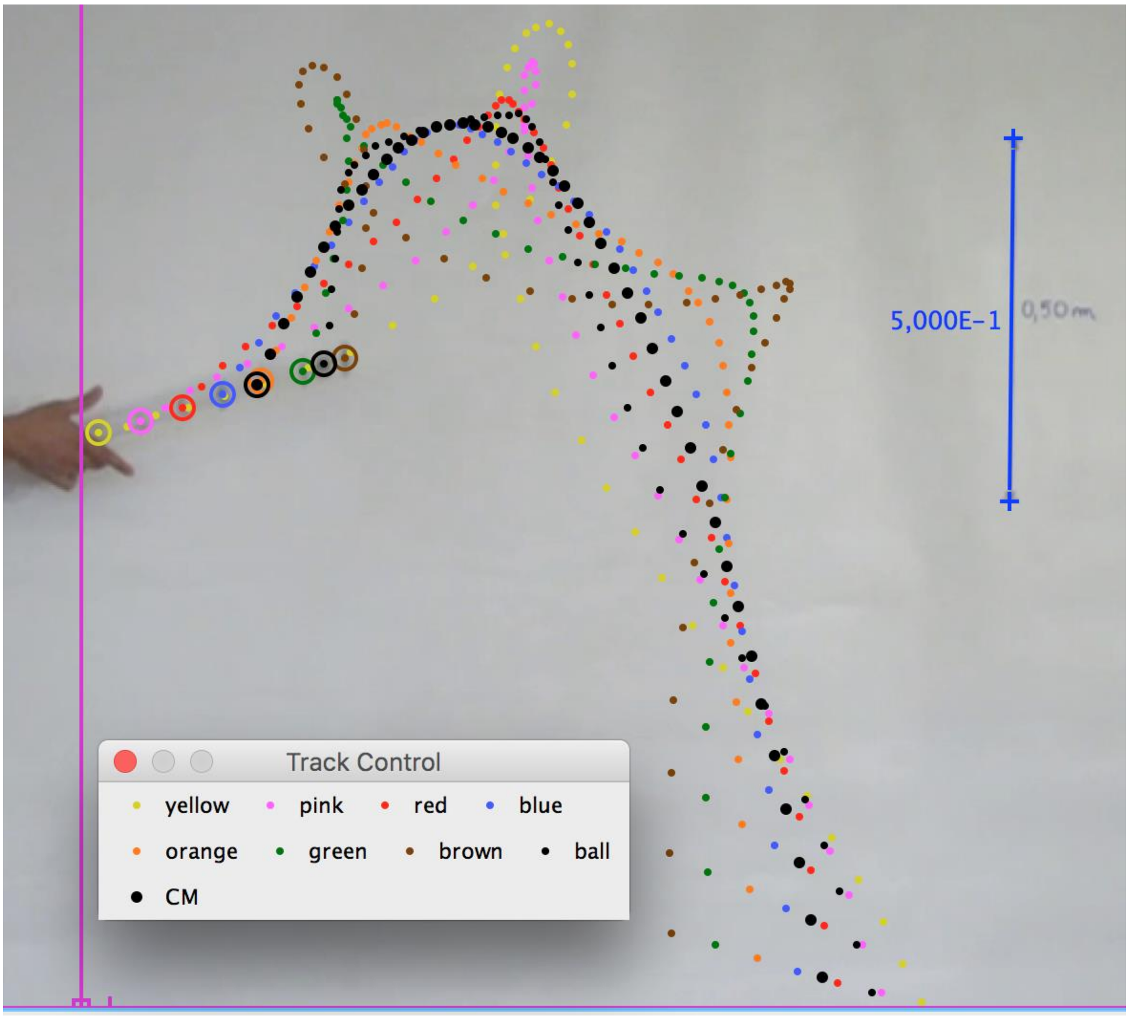


Figure 5: Trajectories of the colour markers of the body and the metallic ball. The black dots correspond to the experimental CM as computed by Tracker software using the know masses of the tube and the metallic ball, and describes a parabolic trajectory.

Figure 6 shows the experimental $x$ and $y$ positions of the CM (from Tracker software) and the corresponding computed CM positions (curve fit from the model) with time. Although the trajectories of the different markers and the ball, shown in figure 5, were very complex, the model managed to fit the experimental data and compute the coordinates of the CM, in a very good agreement with the known position of the body's CM – the average difference between them in $x$ and $y$ coordinates is 3.4 mm and 11.3 mm, respectively.

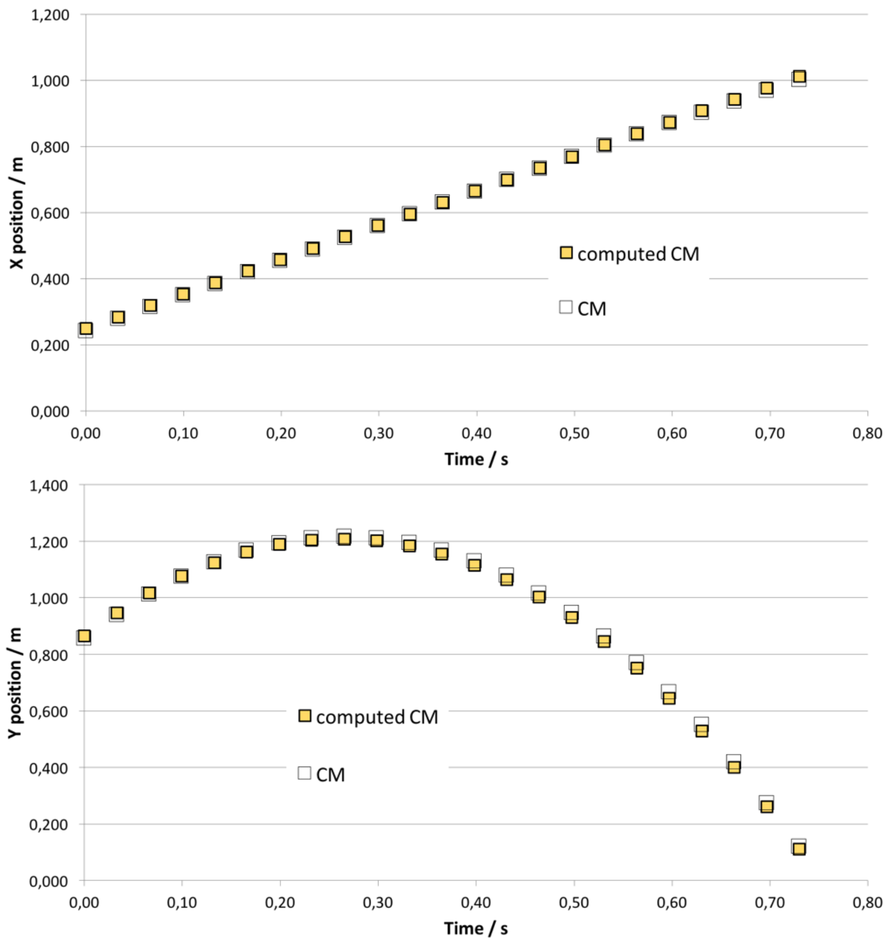


Figure 6: $x$ and $y$ position with time of the computed CM (curve fit) from the model and the experimental CM (from Tracker). The match is very good, within the average experimental error of 3.4 mm and 11.3 mm in $x$ and $y$ coordinates is, respectively.

## 5. Conclusions

In this work we present a computational model to identify the centre of mass of "flying" rigid bodies.

The computational model uses data from video analysis, taken from markers on the body whose positions are fixed between each other during the body's motion. In this study, the markers were separated by 6 cm from each others.

The data computed from the model was compared to the know position of the body's CM. For rigid bodies where the position of the CM is unchanged in the body's frame of reference, the data computed from the model matches quite well the CM's position, with an average experimental accuracy of 2.3 mm and 1.6 mm respectively for the $x$ and $y$ coordinates of the CM. Even when the body's centre of mass position changes in the body's frame of reference during the whole motion, we show that the model still computes the coordinates of the CM in a very good agreement, with an average experimental accuracy of 3.4 mm and 11.3 mm respectively for the $x$ and $y$ coordinates of the CM.

The minimum number of markers to position in a moving body for using the computational model is two, but the accuracy of the model strongly depends on how the

CM moves in the body's reference of frame. In general, the more markers are used for the fitting, the better the results obtained from the computational model. Moreover, these marks on the body should be placed around the region where the CM is expected to move.

The model described in this paper can be used for a system of bodies moving in a plane, for which the CM of each body coincides with its geometric centre. It must be also stressed that the model can also be applied to bodies with unknown masses. The computed CM does not make use of the body's mass, but only the $x$ and $y$ components of the trajectories of the markers in the body.

The computational model can be used in a very large list of contexts. Primarily, in teaching students in higher education level about the importance of the CM's concept for the interpretation of complex motions, such as in robotic systems where equilibrium is essential. In particular, it can be an educational resource for students to investigate how Physics happens in many sports by understanding the motions of athletes, and eventually how to optimize their performances. Teachers may also find useful to let students explore the weird motions that artists and acrobats sometimes do, in their attempts to aggregate science and art.

Studies of complex motions in sports are already being undertaken by us and the results will be soon published elsewhere.

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[11] The video "tube_with_ball_120fps.mp4" can be found at the URL: https://1drv.ms/v/s!AtPtYznj9CaCi-4DiAqQBgpNMPUX6Q